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Polypotentials on a Riemannian manifold

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ABSTRACT

Polyharmonic functions are considered on open sets in a Riemannian manifold R and their potential-theoretic properties are studied using the notion of complete m-potentials. Also one obtains here some characterizations of domains in R on which such complete m-potentials exist.

1. Introduction

A locally-integrable function u on a domain Ω in a Riemannian manifold R is said to be m-harmonic if $\Delta^m u = 0$ in the sense of distributions. When $\Omega = \Re^n$, the m-harmonic functions have nice properties if $n \geqslant 2m+1$. For example, given any m-harmonic function u outside a compact set in $\Re^n$, there exists an m-harmonic function v on $\Re^n$ such that $(u - v)$ tends to zero at infinity if and only if $n \geqslant 2m+1$. We call a domain Ω in a Riemannian manifold R an *m-potential domain* if Ω has certain potential-theoretic properties as $\Re^n$, $n \geqslant 2m+1$. In this note, we prove some intrinsic properties of such m-potential domains.

2. Preliminaries

Let R be a connected, countable, oriented C^∞ Riemannian manifold of dimension n, with local coordinates $x = (x^1, \ldots, x^n)$; dx denotes the volume measure; and the Laplace–Beltrami operator denoted by $\Delta u = div(grad\, u)$ is taken in the sense of distributions.

For any Radon measure $\mu \geqslant 0$ on R, we can construct a superharmonic function s on R such that $(-\Delta)s = \mu$ (see Anandam [3]). In particular, if f is a locally dx-integrable function on R, by considering the positive and the negative parts of the function f, we have a δ-superharmonic function u such that $(-\Delta)u = f$. Again, since u is locally dx-integrable, there exists a δ-superharmonic function v such that $(-\Delta)v = u$ which can be written as $(-\Delta)^2 v = f$. Thus, for any integer $m > 0$, we can construct a δ-superharmonic function g such that $(-\Delta)^m g = f$.

A locally-integrable function u on an open set ω is said to be m-harmonic on ω if $\Delta^m u = 0$. Sometimes we represent an m-harmonic function u in the form $(u_i)_{m \geqslant i \geqslant 1} = (u_m, \ldots, u_1)$ where $u_i = (-\Delta)^{m-i} u$ and $(-\Delta)u_1 = 0$, identifying u with u_m.

Definition 2.1. Let u_i, $1 \leqslant i \leqslant m$, be m locally-integrable functions defined on a domain Ω in R, such that $(-\Delta)u_{j+1} = u_j$, $1 \leqslant j \leqslant m-1$. Write

$$u = (u_i)_{m \geqslant i \geqslant 1} = (u_m, u_{m-1}, \ldots, u_1).$$

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(1) We say that $u = (u_i)$ is a complete m-superharmonic function (respectively a complete m-potential) on Ω if each u_i is superharmonic (respectively potential) on Ω.
(2) If $u = (u_i)$ is a complete m-superharmonic function, the harmonic support of u_1 is referred to as the m-harmonic support of u (written simply as *Supp u*).
(3) If $u = (u_i)$ is such that $(-\Delta)u_1 = 0$, then u is called an m-harmonic function.

Remarks.

(1) The term "*complete m-superharmonic*" is suggested by the terminology completely subharmonic on $\Re^n$ used by Nicolescu [5, p. 16] in a slightly different manner.
(2) If $p = (p_i)$ is a complete m-potential on a domain Ω, then $p_i \equiv 0$ for all i or $p_i > 0$ for all i. In the former case, we write $p = 0$ and in the latter case $p > 0$. When $p > 0$ is a complete m-potential, we refer to p_m as a polypotential of order m or simply as an m-potential. Thus, u is an i-potential if $(u, (-\Delta)u, \ldots, (-\Delta)^{i-1}u)$ is a complete i-potential.
(3) An m-harmonic function $u = (u_i)$ is in general not a complete m-superharmonic function, whereas a positive m-harmonic function is. In fact, by the minimum principle, an m-harmonic function $u = (u_i)_{m \geqslant i \geqslant 1}$ on an open set ω is a complete m-superharmonic function if and only if $\lim_{x \to y} \inf u_i(x) \geqslant 0$, for all $y \in \partial\omega$, the boundary of ω in the Alexandrov compactification of Ω, and $1 \leqslant i \leqslant m-1$.
(4) If $s = (s_i)_{m \geqslant i \geqslant 1}$ is a complete m-superharmonic function on Ω, $s_i \geqslant 0$ for $1 \leqslant i \leqslant m-1$. Hence we say that s is lower bounded (respectively positive) if and only if s_m is lower bounded (respectively positive) in Ω.

3. *m*-potential domains

In this section, we study the properties of positive complete m-superharmonic functions defined on a domain Ω in a Riemannian manifold R, such as the Riesz decomposition, the domination principle, the m-harmonic extension along a closed polar set, the balayage, and the solution to the Riquier problem.

Theorem 3.1. *If a complete m-superharmonic function $s = (s_i)_{m \geqslant i \geqslant 1}$ is positive, that is, $s_m \geqslant 0$, then it is the unique sum of a complete m-potential $p = (p_i)$ and an m-harmonic function $H = (H_i) \geqslant 0$, such that $s_i = p_i + H_i$ for each i.* (*We write $s = p + H$.*)

Proof. Remark that for any complete m-superharmonic function $u = (u_i)_{m \geqslant i \geqslant 1}$, we have $u_i \geqslant 0$, $1 \leqslant i \leqslant m-1$. Hence we write $u \geqslant 0$, if and only if $u_m \geqslant 0$. Since s is a complete m-superharmonic function $\geqslant 0$, $(-\Delta)^k s_m \geqslant 0$ for all $0 \leqslant k \leqslant m$. Suppose for some k, $(-\Delta)^k s_m(x) = 0$ for some point x in Ω. Then $(-\Delta)^k s_m \equiv 0$, that is, s is m-harmonic on Ω.

Let us consider the other case where $(-\Delta)^k s_m > 0$ on Ω for all k, $0 \leqslant k \leqslant m$. Let $s_i = q_i + h_i$ where q_i is a potential and $h_i \geqslant 0$ is harmonic. Write $p_1 = q_1$ and $h_1 = H_1$. Let $(-\Delta)u_1 = p_1$ and $(-\Delta)v_1 = H_1$. Then u_1 and v_1 are superharmonic functions. Since $-\Delta(u_1 + v_1) = s_1 = (-\Delta)s_2$, $u_1 + v_1 = s_2 +$ (a harmonic function). Note that u_1 has a subharmonic minorant since v_1 is superharmonic and $s_2 > 0$. Hence u_1 is a potential p_2 up to an additive harmonic function. Now it is easy to verify $p_2 \leqslant q_2 \leqslant s_2$ and $s_2 - p_2 = H_2 = v_1 +$ (a harmonic function). Consequently, $s_2 = p_2 + H_2$ where $H_2 \geqslant 0$ and $(-\Delta)H_2 = H_1$. Let us repeat the procedure as follows: Let $(-\Delta)u_2 = p_2$ and $(-\Delta)v_2 = H_2$. Since $-\Delta(u_2 + v_2) = s_2 = (-\Delta)s_3$, we can find a potential $p_3 \leqslant q_3 \leqslant s_3$ such that $s_3 - p_3 = H_3 = v_2 +$ (a harmonic function). Consequently, $s_3 = p_3 + H_3$ where $(-\Delta)H_3 = (-\Delta)v_2 = H_2$.

Thus, for $1 \leqslant i \leqslant m$, $s_i = p_i + H_i$ where p_i is a potential, $H_i \geqslant 0$, $(-\Delta)p_{j+1} = p_j$ and $(-\Delta)H_{j+I} = H_j$ for $1 \leqslant j \leqslant m-1$. Consequently, $p = (p_i)$ is a complete m-potential and $H = (H_i)$ is an m-harmonic function and we arrive at the stated decomposition $s = p + H$. □

The uniqueness of decomposition follows from the following proposition.

Proposition 3.2. *Let h be an m-harmonic function and p be a complete m-potential defined on a domain in R. Suppose $h \leqslant p$. Then $h \leqslant 0$.*

Proof. Let $h = (h_i)$ and $p = (p_i)$. Since h_1 is harmonic $\leqslant p_1$, we have $h_1 \leqslant 0$. Then $(-\Delta)h_2 = h_1 \leqslant 0$; hence h_2 is subharmonic $\leqslant p_2$ so that $h_2 \leqslant 0$. Continuing thus, we find $(-\Delta)h_m = h_{m-1} \leqslant 0$; hence h_m is subharmonic $\leqslant p_m$ so that $h_m \leqslant 0$. □

Corollary 1. *If a complete m-superharmonic function $s = (s_i)$ has an m-harmonic minorant $h = (h_i)$, then s has the greatest m-harmonic minorant H.*

Proof. $s - h$ is a positive complete m-superharmonic function. Hence by Theorem 3.1, $s - h = p + h_1$ which implies that $H = h + h_1 \leqslant s$. This function H is the *g.h.m.* of s. For, if an m-harmonic function $h_2 \leqslant s = p + h_1 + h = p + H$, then by the above proposition, $h_2 - H \leqslant 0$. □

Corollary 2. *Let u be a complete m-superharmonic function and v be a complete m-subharmonic function (that is, $-v$ is a complete m-superharmonic function) such that $u \geqslant v$. Then u has the greatest m-harmonic minorant H such that $u \geqslant H \geqslant v$.*

Proof. Let $u = (u_i)$ and $v = (v_i)$, $m \geqslant i \geqslant 1$. Note $u_j \geqslant 0$ and $v_j \leqslant 0$ for all $1 \leqslant j \leqslant m-1$. Since $u_m \geqslant v_m$, there exists a harmonic function h_1 such that $u_m \geqslant h_1 \geqslant v_m$. Then $H_1 = (h_1, 0, \dots, 0)$ is m-harmonic such that $u \geqslant H_1 \geqslant v$. Since $u \geqslant H_1$, by the above corollary, $u - H_1$ has the greatest m-harmonic minorant $H_2 \geqslant 0$ on Ω. Let $H = H_2 + H_1$. Then $u \geqslant H \geqslant v$ on Ω. Also if H_3 is m-harmonic such that $H_3 \leqslant u$, then $H_3 - H_1 \leqslant H_2$, that is, $H_3 \leqslant H_1 + H_2 = H$. □

Corollary 3. *There is no complete m-potential > 0 on Ω if and only if every lower bounded complete m-superharmonic function is m-harmonic.*

Proof. Recall that we say that a complete m-superharmonic function $s = (s_i)_{m \geqslant i \geqslant 1}$ is lower bounded if and only if s_m is lower bounded. Suppose $s_m \geqslant \lambda$. Then $s' = (s'_i)_{m \geqslant i \geqslant 1}$ is positive where $s'_m = s_m - \lambda$ and $s'_i = s_i$ for $1 \leqslant i \leqslant m-1$. If there is no complete m-potential on Ω, then by the above theorem s' and hence s are m-harmonic. The converse is simple. □

Definition 3.3. A domain Ω in R is said to be an m-potential domain if there exists a complete m-potential $(p_i)_{m \geqslant i \geqslant 1} = p > 0$ on Ω.

Remarks.

(1) Suppose there exists a complete m-superharmonic function $(s_i)_{m \geqslant i \geqslant 1} = s \geqslant 0$ on a domain Ω in R, such that $Supp\, s \neq \phi$ (recall that the harmonic support of the superharmonic function s_1 is known as $Supp\, s$, the m-harmonic support of s). Then Ω is an m-potential domain (see the proof of Theorem 3.1). Consequently, Ω is not an m-potential domain if and only if every complete m-superharmonic function on Ω, majorizing an m-harmonic function, is m-harmonic.
(2) Let $p_m(x) = |x|^{2m-n}$ on $\Re^n$, $n \geqslant 2m+1$. Let $(-\Delta)^i p_m = p_{m-i}$, $1 \leqslant i \leqslant m-1$. Then $p = (p_m, \dots, p_1)$ is an m-potential on $\Re^n$. Hence, for any $n \geqslant 2m+1$, $\Re^n$ is an m-potential domain.
(3) It is shown in Anandam [3] that if u is a superharmonic function on $\Re^n$, $2 \leqslant n \leqslant 2m$, such that $(-\Delta)^i u \geqslant 0$ for $0 \leqslant i \leqslant m$, then u is a constant. Consequently, $\Re^n$ is not an m-potential domain if $2 \leqslant n \leqslant 2m$. (See Corollary 3 to Theorem 4.5.)

Proposition 3.4. *In an m-potential domain Ω, let $s = (s_i)_{m \geqslant i \geqslant 1}$ be a complete m-superharmonic function $\geqslant 0$ and $p = (p_i)_{m \geqslant i \geqslant 1}$ be a complete m-potential. Suppose $s_1 \geqslant p_1$. Then $s \geqslant p$ on Ω.*

Proof. Let $(-\Delta)u = s_1 - p_1$. Then u is superharmonic on Ω. Since $(-\Delta)s_2 = s_1$ and $(-\Delta)p_2 = p_1$, we have $s_2 = p_2 + u +$ (a harmonic function). Consequently, u has a subharmonic minorant and hence is the sum of a potential q and a harmonic function. Then $p_2 + q \leqslant s_2$ which shows that $s_2 \geqslant p_2$. Proceeding thus, we finally arrive at the conclusion $s_i \geqslant p_i$ for all i, $1 \leqslant i \leqslant m$. That is, $s \geqslant p$. □

Corollary *(Domination principle for m-potentials). In an m-potential domain Ω, let $p = (p_i)_{m \geqslant i \geqslant 1}$ be a complete m-potential, such that p_1 is locally bounded. Let $s = (s_i)_{m \geqslant i \geqslant 1}$ be a complete m-superharmonic function $\geqslant 0$ such that $s_1 \geqslant p_1$ on $Supp\, p$. Then $s \geqslant p$ on Ω.*

Theorem 3.5 *(Riquier problem). (See Nicolescu [5, p. 28].) Let ω be a regular domain in Ω. Then, given m continuous functions f_i on $\partial\omega$, there exists a unique m-harmonic function $h = (h_i)_{m \geqslant i \geqslant 1}$ on ω such that $h \to f = (f_i)$ on $\partial\omega$.*

Proof. Let h_1 be harmonic on ω tending to f_1 on $\partial\omega$. Extend this function outside $\bar{\omega}$ so as to get a continuous function h_1^* on Ω with compact support. Let $(-\Delta)u_2 = h_1^*$ on Ω. Write $h_2 = u_2 + H^{\omega}_{f_2 - u_2}$ on ω. Then $h_2 \to f_2$ on $\partial\omega$ and $(-\Delta)h_2 = h_1$ on ω. Repeating this procedure we obtain an m-harmonic function $h = (h_i)_{m \geqslant i \geqslant 1}$ on ω such that $h_i \to f_i$ on $\partial\omega$ for each i. □

Theorem 3.6. *Let e be a closed locally-polar set in an open set ω in R. Suppose u is an m-harmonic function on $\omega \setminus e$, locally bounded on ω. Then u extends uniquely as an m-harmonic function on ω.*

Proof. Let $h = (h_i)_{m \geqslant i \geqslant 1}$ be an m-harmonic function on $\omega \setminus e$, such that each h_i is locally bounded on ω. Now h_i is harmonic on $\omega \setminus e$, locally bounded on ω. Hence it extends as a harmonic function h_1 on ω. Let $(-\Delta)H_1 = h_1$. Then on $\omega \setminus e$, there exists a harmonic function u such that $H_1 = h_2 + u$. Since H_1 is finite continuous on ω and h_2 is locally bounded on ω by hypothesis, the harmonic function u extends as a harmonic function on ω. Hence h_2 extends continuously as h_2 such that $(-\Delta)h_2 = h_1$ on ω.

Thus proceeding, we finally arrive at an m-harmonic function $h' = (h'_i)_{m \geqslant i \geqslant 1}$ on ω such that $h'_i = h_i$ on $\omega \setminus e$. The uniqueness on extension follows from the uniqueness of extension of h_1 as a harmonic function on ω. □

Remark. As above, it can be proved also: Let e be a closed locally-polar set in ω. Let s be a complete m-superharmonic function on $\omega \setminus e$, locally bounded on a neighborhood of e. Then s extends as a complete m-superharmonic function on ω. However, note that if the complete m-superharmonic function s defined on $\omega \setminus e$ is only locally lower bounded on ω, then s need not extend as a complete m-superharmonic function on ω. For example, consider the complete 2-superharmonic function $s = (s_2, s_1)$ where s_1 and s_2 are superharmonic functions on $\Re^3 \setminus \{0\}$ and $s_1 = 2|x|^{-1}$, $s_2|x|^{-1} = -|x| + |x|^{-1}$.

Theorem 3.7 *(Balayage). Let $s > 0$ be a complete m-superharmonic function on an m-potential domain Ω in a Riemannian manifold R. Let $e \subset \Omega$. Then there exists a complete m-superharmonic function $q > 0$ on Ω such that $q \leqslant s$ on Ω, q is m-harmonic on $\Omega \setminus \bar{e}$, and $s = q + h$ on e° where h is $(m-1)$-harmonic on e°. If $(Q_i)_{m \geqslant i \geqslant 1} = Q > 0$ is any complete m-superharmonic function on Ω such that $Q_1 \geqslant s_1$ on e, then $Q \geqslant q$ on Ω.*

Proof. Let $s = (s_i)_{m \geqslant i \geqslant 1}$ be the complete m-superharmonic function on Ω. Let $q_1 = \hat{R}^e_{s_1}$ on Ω. Let $(-\Delta)u_2 = q_1$ and $(-\Delta)v_2 = s_1 - q_1$ so that u_2 and v_2 are superharmonic on Ω, and $u_2 + v_2 = s_2 +$ (a harmonic function). Hence u_2 is a potential q_2 up to an additive harmonic function and $q_2 \leqslant s_2$ on Ω. Moreover, since $s_1 - q_1 = 0$ on e°, v_2 is harmonic on e°; and on $\Omega \setminus \bar{e}$, since q_1 is harmonic, u_2 (and hence q_2) is 2-harmonic. Thus, q_2 is a potential on Ω, $q_2 \leqslant s_2$, q_2 is 2-harmonic on $\Omega \setminus \bar{e}$, and $s_2 = q_2 +$ (a harmonic function h_2) on e°; also $(-\Delta)q_2 = q_1$ on Ω.

Let $(-\Delta)u_3 = q_2$ and $(-\Delta)v_3 = s_2 - q_2$ and proceed above to obtain a potential q_3 on Ω, $q_3 \leqslant s_3$, q_3 is 3-harmonic on $\Omega \setminus \bar{e}$, and $s_3 = q_3 +$ (a 2-harmonic function h_3) on e°; also $(-\Delta)q_3 = q_2$. In this manner, we can go far to construct a potential $q_m > 0$ on Ω with similar properties, namely: $q_m \leqslant s_m$, q_m is m-harmonic on $\Omega \setminus \bar{e}$, and $s_m = q_m +$ (an $(m-1)$-harmonic function h_m) on e°; and $(-\Delta)q_m = q_{m-1}$ on Ω.

Set now $q = (q_i)_{m \geqslant i \geqslant 1}$ and $h = (h_m, \dots, h_2, 0)$. Then q is a complete m-superharmonic function > 0 on Ω, q is m-harmonic on $\Omega \setminus \bar{e}$, h is $(m-1)$-harmonic on e°, and $s = q + h$ on e°. (Note that q is a complete m-potential if e is relatively compact, or if s is a complete m-potential on Ω.)

Finally, if $(Q_i)_{m \geqslant i \geqslant 1} = Q \geqslant 0$ is a complete m-superharmonic function on Ω such that $Q_1 \geqslant s_1$ on e, then $Q_1 \geqslant \hat{R}^e_{s_1} = q_1$ on Ω. That is, $(-\Delta)Q_2 \geqslant (-\Delta)q_2$. Hence, $Q_2 = q_2 + u_2$, where u_2 is superharmonic on Ω. Since u_2 has a subharmonic minorant on Ω, u_2 is a sum of a potential p_2 and a harmonic function; also q_2 is a potential. Write $Q_2 = P_2 + H_2$, where P_2 is a potential and $H_2 \geqslant 0$ is harmonic, so that $q_2 + p_2 = P_2$. Consequently, $q_2 \leqslant P_2 \leqslant Q_2$ on Ω. Proceeding thus, we show that $q_i \leqslant Q_i$ on Ω for all i, $1 \leqslant i \leqslant m$. That is, $q \leqslant Q$ on Ω. $\square$

Corollary. *Suppose Ω is not an m-potential domain. Then every complete m-superharmonic function $s \geqslant 0$ on Ω is j-harmonic for some j, $0 \leqslant j \leqslant m-1$.*

Proof. (1) Suppose there is no potential on Ω. Then every positive superharmonic function on Ω is a constant, so that if $s \geqslant 0$ is a complete m-superharmonic function on Ω, $s_i = 0$ for $1 \leqslant i \leqslant m-1$; that is, s is a constant for the form $(\lambda, \dots, 0)$, which we term as 0-harmonic.

(2) Suppose there is a complete j-potential on Ω, $1 \leqslant j \leqslant m-1$, but no complete $(j+1)$-potential on Ω. Let $(s_i)_{m \geqslant i \geqslant 1} = s \geqslant 0$ be a complete m-superharmonic function on Ω. Let i be the smallest index such that $s_i > 0$. Then, as in the theorem we can construct a complete $(m-i+1)$-potential $q = (q_m, \dots, q_i)$ on Ω with $q_i = \hat{R}^\omega_{s_i}$ for some relatively compact open set ω. Since by hypothesis there is no complete $(j+1)$-potential on Ω, $m-i+1 < j+1$ and hence $i > m-j \geqslant 1$. Since $s = (s_m, \dots, s_i, 0, \dots, 0)$, s is $(m-i+1)$-harmonic on Ω; since $m-i+1 \leqslant j$, s is j-harmonic on Ω. $\square$

4. Some characterizations of *m*-potential domains

In this section, we obtain some necessary and sufficient conditions for a domain Ω in R to be an m-potential domain and give a representation for the m-harmonic functions defined outside a compact set in an m-potential domain.

Lemma 4.1. *Suppose Ω is an m-potential domain in a Riemannian manifold R. Let $Q = (Q_i)$ be any given complete m-potential > 0 on Ω. Then given any potential $p_1 > 0$ on Ω with compact harmonic support, there exists a complete m-potential $p = (p_i)_{m \geqslant i \geqslant 1}$ such that for some $\lambda > 0$ and each i, $p_i \leqslant \lambda Q_i$ near infinity, that is, outside a compact set in Ω.*

Proof. Since p_1 is a potential with compact harmonic support, we can find λ_1 such that $p_1 \leqslant \lambda_1 Q_1$ outside a compact set in Ω. Let $(-\Delta)u = p_1$ and $(-\Delta)v = \lambda_1 Q_1 - p_1$ on Ω. Then u is superharmonic on Ω, v is superharmonic near infinity and $(-\Delta)(u+v) = \lambda_1 Q_1 = (-\Delta)(\lambda_1 Q_2)$. Hence $(u+v) = \lambda_1 Q_2 +$ (harmonic function) on Ω, so that u has a subharmonic minorant outside a compact set. Write then $u = p_2 + h$ where p_2 is a potential on Ω and h is (not necessarily positive) harmonic on Ω.

Thus, p_2 is such that $(-\Delta)p_2 = p_1$ and $p_2 = \lambda_1 Q_2 +$ (a subharmonic function) outside a compact set in Ω. Since a subharmonic function outside a compact set is of the form $s(x) + q(x)$ near infinity where s is subharmonic on Ω and q is a potential with compact harmonic support on Ω (an easy extension of [1, Théorème 1]; for this result in $\Re^n$, $n \geqslant 3$, see [2, Theorem 1]), we have $p_2 = \lambda_1 Q_2 + s + q$ near infinity. Since $s \leqslant p_2$ near infinity, $s \leqslant 0$ on Ω; and since q is a potential with compact support, $q \leqslant \alpha_1 Q_2$ near infinity for some $\alpha_1 > 0$. Thus, if $\lambda_2 = \lambda_1 + a_1$, $p_2 \leqslant \lambda_2 Q_2$ near infinity.

We repeat the above procedure to obtain a potential p_3 on Ω such that $(-\Delta)p_3 = p_2$ on Ω and $p_3 \leqslant \lambda_3 Q_3$ near infinity. Continuing thus we arrive at $p = (p_i)_{m \geqslant i \geqslant 1}$ which is a complete m-potential on Ω such that $p_i \leqslant \lambda Q_i$ near infinity for $\lambda = \max_{1 \leqslant i \leqslant m} \lambda_i$. □

Note. In the above lemma, if p_1 is a finite continuous potential with compact harmonic support, then in $p = (p_i)_{m \geqslant i \geqslant 1}$, all the potentials p_i are finite continuous and for some $\lambda > 0$, $p_i \leqslant \lambda Q_i$ on the whole of Ω.

Proposition 4.2. *Let Ω be an m-potential domain in a Riemannian manifold. Then every subdomain ω of Ω is an m-potential domain.*

Proof. Let $Q = (Q_i)_{m \geqslant i \geqslant 1}$ be a complete m-potential on Ω. Let q_i be the potential on ω such that $Q_i = q_i +$ (a positive harmonic function h_i) on ω.

Let $p_1 > 0$ be a finite continuous potential such that $p_1 \leqslant Q_1$ on ω. Let $(-\Delta)s_1 = p_1$ on ω. Since $(-\Delta)Q_2 = Q_1$ on Ω by hypothesis, $\Delta(s_1 - Q_1) \geqslant 0$ so that $s_1 - Q_2 = t_1$ is subharmonic on ω. Hence on ω, s_1 has a subharmonic minorant and t_1 has a superharmonic majorant. Let $s_1 = p_2 + H_2$ and $t_1 = -(p_2' + H_2')$ be the Riesz decompositions, so that $(p_2 + p_2') + (H_2 + H_2') = q_2 + h_2$ on ω. We deduce that $p_2 + p_2' = q_2$ on ω. Consequently, $(-\Delta)p_2 = (-\Delta)s_1 = p_1$ on ω and $p_2 \leqslant q_2 \leqslant Q_2$ on ω.

Proceeding similarly, by recurrence as in the above lemma, we can construct a complete m-potential $p = (p_m, \ldots, p_2, p_1)$ on ω such that $p_i \leqslant Q_i$ on ω. Hence ω is an m-potential domain. □

Proposition 4.3. *Let Ω be an m-potential domain in Riemannian manifold. Then any complete m-superharmonic function with compact m-harmonic support is the unique sum of a complete m-potential and an m-harmonic function.*

Proof. Let $s = (s_i)_{m \geqslant i \geqslant 1}$ be a complete m-superharmonic function such that s_1 has compact harmonic support. If s_1 is harmonic, the proposition is trivial. Otherwise, write $s_1 = p_1 + h_1$ where p_1 is a potential with compact harmonic support and h_1 is a (not necessarily positive) harmonic function on Ω. Then by the above lemma there exists a complete m-potential $p = (p_i)_{m \geqslant i \geqslant 1}$.

Since $(-\Delta)s_2 = s_1$ and $(-\Delta)p_2 = p_1$, if $(-\Delta)H_2 = h_1$, then $s_2 = p_2 + H_2 +$ (a harmonic function f_2). Write $h_2 = H_2 + f_2$. Then $s_2 = p_2 + h_2$, where $(-\Delta)h_2 = h_1$. Proceeding thus, we arrive at the equation $s_i = p_i + h_i, 1 \leqslant i \leqslant m$, where $(-\Delta)h_{j+1} = h_j$ for $1 \leqslant j \leqslant m-1$; that is, $h = (h_i)_{m \geqslant i \geqslant 1}$ is an m-harmonic function such that $s = p + h$ on Ω. □

The uniqueness of decomposition follows from Proposition 3.2.

Theorem 4.4. *Let Ω be an m-potential domain. Let $Q = (Q_i)_{m \geqslant i \geqslant 1}$ be a given complete m-potential on Ω. Then given an m-harmonic function u outside a compact set in Ω, there exist i-potentials P_i, P_i' on Ω, $m \geqslant i \geqslant 1$ (see Remark 2 following Definition 2.1 for the meaning of an i-potential) and an m-harmonic function v on the whole of Ω, such that $u = \sum_{i=1}^{m}(P_i - P_i') + v$ outside a compact set in Ω; moreover, for some $\lambda > 0$, P_i, P_i' are majorized by λQ_i on Ω.*

Proof. Since $\Delta^{m-1}u$ is harmonic outside a compact set, there exist finite continuous potentials q_1, q_1' and a harmonic function h on Ω such that $\Delta^{m-1}u = q_1 - q_1' + h$ outside a compact set in Ω. Then by Lemma 4.1, there exist m-potentials (q_i), (q_i'), $m \geqslant i \geqslant 1$, such that $(-\Delta)^{m-1}q_m = q_1$ and $(-\Delta)^{m-1}q_m' = q_1'$; also, q_i, q_i' both are majorized by λQ_i for every i. Choose h_m so that $\Delta^{m-1}h_m = h_1$ on Ω.

Now, if m is even we shall write $P_m = q_m'$ and $P_m' = q_m$; and if m is odd we shall write $P_m = q_m$ and $P_m' = q_m'$ so that we can write $\Delta^{m-1}u = \Delta^{m-1}(P_m - P_m' + h_m)$. Hence, $u = P_m - P_m' + h_m +$ (an $(m-1)$-harmonic function u') near infinity. By recurrence we arrive at the representation $u = (P_m - P_m') + \cdots + (P_1 - P_1') + (h_m + \cdots + h_1)$ near infinity. Moreover, we have shown above P_m, P_m' are majorized by λQ_m and similarly P_i, P_i' are majorized by λQ_i for every i. □

Corollary 1. *Let Ω be an m-potential domain. Let $(Q_i)_{m \geqslant i \geqslant 1}$ be a given complete m-potential on Ω. Then given any m-harmonic function u outside a compact set, there exists an m-harmonic function v on Ω such that $|(-\Delta)^j(u - v)| \leqslant \lambda \sum_{i=1}^{m-1} Q_i$ for some $\lambda > 0$, $0 \leqslant j \leqslant m-1$.*

Corollary 2. *Let Ω be an m-potential domain. Let u be defined outside a compact set in Ω such that $(-\Delta)^m u \geqslant 0$. Then there exist v on Ω, $(-\Delta)^m v \geqslant 0$ and i-potentials P_i, P_i' on Ω as in the theorem above such that $u = v + \sum_{i=1}^{m}(P_i - P_i')$ near infinity.*

Proof. (1) First we prove an auxiliary result in a Riemannian manifold R: Suppose y is a point fixed in R. Let $G(y, x)$ be the Green potential (respectively the Evans potential, see Nakai [4]) on R if it is hyperbolic (respectively parabolic). Then, given any superharmonic function u_1 outside a compact set in R, there exist a superharmonic function v_1 on R and a constant $\alpha \geqslant 0$ such that $u_1 = v_1 - \alpha G_y$ near infinity.

Let u_1 be defined outside a compact set A in R. Let k_0 be an outer regular compact set and ω_0 be a regular domain (for the Dirichlet problem) such that $\{y\} \cup A \subset \mathring{k}_0 \subset k_0 \subset \omega$. Let us replace u_1 in $\omega_0 \setminus k_0$ by the Dirichlet solution with boundary values u_1. Thus we can consider the superharmonic function u_1 on $R \setminus A$ is harmonic on $\omega_0 \setminus k_0$. Let k be an outer regular compact set and ω be a regular domain such that $k_0 \subset \mathring{k} \subset k \subset \omega \subset \bar{\omega} \subset \omega_0$. Let Df denote the Dirichlet solution on ω with boundary value f.

Let $s(x) = G(y, x)$. Since s is superharmonic on ω with harmonic support $\{y\}$, $s > Ds$ on ω. Hence we can choose $\alpha \geqslant 0$ so that $-\alpha(s - Ds) \leqslant u_1 - Du_1$ on ∂k. Hence, by the maximum principle, $-\alpha(s - Ds) \leqslant u_1 - Du_1$ on $\omega \setminus k$. If we define

$$v_1 = \begin{cases} u_1 + as & \text{on } R \setminus \omega, \\ D(u_1 + as) & \text{on } \omega, \end{cases}$$

v_1 is superharmonic on R and $u_1 = v_1 - \alpha s = v_1 - \alpha G_y$ on $R \setminus \omega$.

(2) Consider now u on Ω for which $(-\Delta)^m u \geqslant 0$ near infinity. Since $u_1 = (-\Delta)^{m-1} u$ is superharmonic, by (1), there exists v_1 superharmonic on R such that $u_1 = v_1 - \alpha G_y$ where $G_y(y) = G(x, y)$ is the Green function of Ω with pole $\{y\}$. Choose v and Q on Ω such that $(-\Delta)^{m-1} v = v_1$ and $(-\Delta)^{m-1} Q = \alpha G_y$. Then $u = v - Q + u'$ near infinity where $(-\Delta)^{m-1} u' = 0$, so that $(-\Delta)^m (Q - u') = 0$ near infinity. Now apply the above theorem to obtain the representation for u near infinity, as stated in the corollary. $\square$

Theorem 4.5. *Let Ω be a domain in R, carrying the Green function $G(x, y)$. Then Ω is an m-potential domain if and only if there exist two points x_0 and x_1 in Ω such that $\int_{\Omega^{m-1}} G(x_1, y_1) G(y_1, y_2) \cdots G(y_{m-2}, y_{m-1}) G(y_{m-1}, x_0)\, dy_1\, dy_2 \cdots dy_{m-1} < \infty$.*

Proof. (1) Let Ω be an m-potential domain with a complete m-potential $Q = (Q_i)_{m \geqslant i \geqslant 1}$. Choose a compact set k in Ω such that $p_1 = \hat{R}_1^k > 0$ is a finite continuous potential on Ω. Then by Lemma 4.1, there exists an m-potential $p = (p_i)_{m \geqslant i \geqslant 1}$ on Ω such that $p_i \leqslant \lambda Q_i$. Since p_2 is a potential on Ω such that $\Delta p_2 = -p_1 - \hat{R}_1^k$,

$$p_2(x) = \int_{\Omega} G(x, y) \hat{R}_1^k(y)\, dy.$$

Similarly,

$$p_3(x) = \int_{\Omega} G(x, y_1) p_2(y_1)\, dy_1 = \int_{\Omega} G(x, y_1) G(y_1, y) \hat{R}_1^k(y)\, dy\, dy_1.$$

Thus proceeding,

$$p_m(x) = \int_{\Omega} G(x, y_1) G(y_1, y_2) \cdots G(y_{m-2}, y) \hat{R}_1^k(y)\, dy\, dy_1 \cdots dy_{m-2}.$$

Since $p_m(x)$ is a potential on Ω, $p_m(x_1)$ is finite for some x_1 in Ω. Hence,

$$\int_{\Omega} \left[\int_{\Omega^{m-2}} G(x_1, y_1) \cdots G(y_{m-2}, y)\, dy_1 \cdots dy_{m-2} \right] \hat{R}_1^k(y)\, dy < \infty.$$

We recall that from [6], it can be seen that given a measure μ on Ω, $\int_{\Omega} G(x, y)\, d\mu(y)$ is a potential if and only if for one (and hence any) nonpolar compact k, $\int \hat{R}_1^k(y)\, d\mu(y) < \infty$. Using this, we conclude that

$$u(x) = \int_{\Omega} G(x, y) \left[\int_{\Omega^{m-2}} G(x_1, y_1) \cdots G(y_{m-2}, y)\, dy_1 \cdots dy_{m-2} \right] dy$$

is a potential on Ω. Hence for some $x_0 \in \Omega$, $u(x_0) < \infty$. That is,

$$\int_{\Omega^{m-1}} \left[G(x_1, y_1) \cdots G(y_{m-2}, y_{m-1}) \right] G(y_{m-1}, x_0)\, dy_1 \cdots dy_{m-2}\, dy_{m-1} < \infty.$$

(2) Conversely, suppose the integral is finite. Then $Q_2(x) = \int_{\Omega} G(x, y) G(x_0, y)\, dy \neq \infty$ everywhere; hence Q_2 is a potential on Ω such that $\Delta Q_2(x) = -G(x_0, x) = -Q_1(x)$. Again, since $\int_{\Omega} G(x, y_1) Q_2(y_1)\, dy_1 = \int_{\Omega \times \Omega} G(x, y_1) G(y_1, y_2) \times G(y_2, x_0)\, dy_1\, dy_2$ is finite at some point, we have a potential $Q_3(x)$ on Ω such that $\Delta Q_3 = -Q_2$.

Thus proceeding, we finally arrive at the situation where we have a potential $Q_{m-1}(x)$ on Ω such that $\Delta Q_{m-1} = -Q_{m-2}$. Hence $Q_m(x) = \int_{\Omega} G(x, y) Q_{m-1}(y)\, dy$ is a hyperharmonic function on Ω. But $Q_m(x_1) < \infty$ by hypothesis hence Q_m also is a potential on Ω and $\Delta Q_m = -Q_{m-1}$. Consequently, $Q = (Q_i)_{m \geqslant i \geqslant 1}$ is a complete m-potential on Ω. $\square$

Corollary 1. *Let Ω be a domain in R carrying the harmonic Green function $G(x,y)$. Then, Ω is an m-potential domain if and only if for one (and hence for any) nonpolar compact set $k \subset \Omega$:*

$$\int \hat{R}_1^k(y_1)G(y_1,y_2)\cdots G(y_{m-2},y_{m-1})\hat{R}_1^k(y_{m-1})\,dy_1\cdots dy_{m-1} < \infty.$$

Proof. As mentioned above, for any Radon measure μ, $\int_\Omega G(x,y)\,d\mu(y)$ is a potential if and only if for one (and hence any) nonpolar compact set k in Ω, $\int_\Omega \hat{R}_1^k\,d\mu(y)$ is finite.

(1) Let Ω be an m-potential domain. Then by the above theorem, there exist two points x_0 and x_1 in Ω such that

$$\int_\Omega G(x_1,y_1)G(y_1,y_2)\cdots G(y_{m-1},x_0)\,dy_1\cdots dy_{m-1} < \infty,$$

which implies that

$$\int_\Omega G(x,y_1)\left[\int_{\Omega^{m-2}} G(y_1,y_2)\cdots G(y_{m-1},x_0)\,dy_2\cdots dy_{m-2}\right]dy_1$$

is a potential on Ω.

Hence for any nonpolar compact set k,

$$\int_{\Omega^{m-1}} \hat{R}_1^k(y_1)G(y_1,y_2)\cdots G(y_{m-1},x_0)\,dy_1\cdots dy_{m-1} < \infty,$$

which implies that

$$\int G(x,y_{m-1})\left[\int \hat{R}_1^k(y_1)G(y_1,y_2)\cdots G(y_{m-2},y_{m-1})\,dy_1\cdots dy_{m-2}\right]dy_{m-1}$$

is a potential so that

$$\int \hat{R}_1^k(y_{m-1})\hat{R}_1^k(y_1)G(y_1,y_2)\cdots G(y_{m-2},y_{m-1})\,dy_1\cdots dy_{m-1} < \infty.$$

(2) The converse follows by retracing the above steps. □

Corollary 2. *A domain Ω in R is an m-potential domain if and only if for some (and hence any) z in Ω, there exists a potential u_z on Ω such that $(-\Delta)^m u_z = \delta_z$.*

Proof. Suppose there exists a complete m-potential on Ω. Then for any $z \in \Omega$, $p_1(x) = G(z,x)$ is a potential with point support in Ω. Hence by Lemma 4.1, there exists a complete m-potential $p = (p_m,\dots,p_1)$ on Ω. Set $u = p_m$. Then u is a potential on Ω such that $(-\Delta)^{m-1}u(x) = p_1(x) = G(z,x)$, so that $(-\Delta)^m u = \delta_z$.

Conversely, let u be a potential on Ω such that $(-\Delta)^m u = \delta_z$ for some $z \in \Omega$. Then in the sense of distributions we have the following equalities:

$$\begin{aligned}
&(-\Delta)^{m-1}u(x) = G(z,x) + h_1(x), \quad h_1 \text{ is harmonic on } \Omega,\\
&(-\Delta)\big[(-\Delta)^{m-2}u(x) - h_2(x)\big] = G(z,x), \qquad (-\Delta)h_2 = h_1,\\
&(-\Delta)^{m-2}u(x) = \int G(x,y_{m-1})G(y_{m-1},z)\,dy_{m-1} + (h_2 + \text{a harmonic function})\\
&\qquad\qquad\quad = \int G(x,y_{m-1})G(y_{m-1},z)\,dy_{m-1} + (-\Delta)h_3,\\
&(-\Delta)^{m-3}u(x) = \int G(x,y_{m-2})G(y_{m-2},y_{m-1})G(y_{m-1},z)\,dy_{m-2}\,dy_{m-1} + (h_3 + \text{a harmonic function}).
\end{aligned}$$

Thus proceeding, we arrive at the equality

$$u(x) = \int G(x,y_1)G(y_1,y_2)\cdots G(y_{m-1},z)\,dy_1\cdots dy_{m-1} + h_m,$$

where h_m is an m-harmonic function on Ω. Since $u(x)$ is a potential, for some $x \in \Omega$,

$$\int G(x,y_1)G(y_1,y_2)\cdots G(y_{m-1},z)\,dy_1\cdots dy_{m-1} < \infty.$$

Hence by the above theorem, Ω is an m-potential domain. □

Corollary 3. *(See [3].)* $\Re^n$ *is an m-potential domain if and only if* $n \geqslant 2m+1$.

Proof. If $n \geqslant 2m+1$, then $u(x) = |x|^{2m-n}$ is a complete m-potential on $\Re^n$. On the other hand, if we suppose that there exists a complete m-potential on $\Re^n$ when $n \leqslant 2m$, by the above corollary there should exist a potential u on $\Re^n$ such that $(-\Delta)^m u = \delta$. But this is not possible. For, in this case u should be of the form $\alpha_n |x|^{2m-n} + \beta_m(x)$ or $\alpha_n |x|^{2m-n} \log|x| + \beta_m(x)$ where $\alpha_n \neq 0$, and $\beta_m(x) = |x|^{2m-2} h_m(x) + \cdots + h_1(x)$ is an m-harmonic function on $\Re^n$. Let $d\rho_\alpha^r(x)$ be the harmonic measure on $|x| = r > a$. Then we should have

$$\int u \, d\rho_\alpha^r(x) = a_n r^{2m-n} \log r + r^{2m-2} h_m(a) + \cdots + h_1(a) \quad \text{if } 2m-n \text{ is even,} \quad \text{and}$$

$$\int u \, d\rho_\alpha^r(x) = a_n r^{2m-n} + r^{2m-2} h_m(a) + \cdots + h_1(a) \quad \text{if } 2m-n \text{ is odd.}$$

But this is not possible. For, if we allow $r \to \infty$, then the left side tends to zero since u is a potential but not the right side since $\alpha_n \neq 0$. This contradiction shows that when $n \leqslant 2m$, $\Re^n$ cannot be an m-potential domain. □

5. Tapered *m*-potential domains

In the special case of $\Re^n$, $n \geqslant 2m+1$, we have a complete m-potential $p = (p_i)_{m \geqslant i \geqslant 1}$ where $p_m(x) = |x|^{2m-n}$ so that for all i, $p_i(x) \to 0$ when $|x| \to \infty$. We do not expect this type of complete m-potentials to exist on a general m-potential domain. So we propose the following definition:

Definition 5.1. An m-potential domain Ω is said to be **tapered** if there exists a complete m-potential $Q = (Q_i)_{m \geqslant i \geqslant 1}$ on Ω such that outside a compact set $Q_i \leqslant M$ for all i and some positive M.

Remarks.

(1) If Ω is a tapered m-potential domain, using the Note following Lemma 4.1, we show that there exists a complete m-potential $p = (p_i)_{m \geqslant i \geqslant 1}$ where each p_i is a bounded continuous potential on Ω.
(2) Every subdomain of a tapered m-potential domain is tapered. (To prove this we use (1) above and the proof of Proposition 4.2.)

Theorem 5.2. *Let Ω be a tapered m-potential domain in R. Then given an m-harmonic function u outside a compact set in Ω, there exists an m-harmonic function v on Ω such that $(u-v)$ is bounded near infinity.*

Proof. From Corollary 1 to Theorem 4.4, we note that there exists an m-harmonic function v on Ω such that $|u - v| \leqslant \lambda \sum_{i=1}^m Q_i$ near infinity. Since Ω is tapered, we can choose a complete m-potential $Q = (Q_i)_{m \geqslant i \geqslant 1}$ such that each Q_i is a bounded continuous potential on Ω. Hence the theorem is proved. □

The following theorem gives a sufficient condition for a domain to be a tapered m-potential domain.

Theorem 5.3. *Suppose s is a bounded function on a domain Ω in R such that $(-\Delta)s = 1$. Then Ω is a tapered m-potential domain.*

Proof. s is a finite continuous superharmonic function on Ω; and by hypothesis it is bounded also. Hence, using the Riesz decomposition $s = p_1 +$ (a harmonic function), we find a bounded continuous potential p_1 on Ω such that $(-\Delta)p_1 = 1$. Suppose $p_1 \leqslant M_1$.

Let $(-\Delta)u = p_1$ and $(-\Delta)v = M_1 - p_1$. Then u and v are finite continuous superharmonic function on Ω such that $u + v = M_1 s +$ (a harmonic function h) on Ω. Since s is bounded, u and v have subharmonic minorants so that $u = p_2 + h_2$ and $v = p_2' + h_2'$ where p_2 and p_2' are finite continuous potential on Ω, h_2 and h_2' are harmonic. Note $p_2 + p_2' = M_1 p_1$ so that p_2 is a bounded continuous potential such that $(-\Delta)p_2 = p_1$ on Ω. Suppose $p_2 \leqslant M_2$.

We repeat the above procedure to obtain a finite continuous potential p_3 on Ω such that $p_3 \leqslant M_2 p_1$ and $(-\Delta)p_3 = p_2$. Suppose $p_3 \leqslant M_3$. Proceeding similarly we prove that for any $i \geqslant 1$, p_i is a bounded continuous potential on Ω such that $(-\Delta)p_{i+1} = p_i$. In particular, $p = (p_i)_{m \geqslant i \geqslant 1}$ is a tapered complete m-potential on Ω. □

Remarks.

(1) The above sufficient condition is not a necessary condition for Ω to be a tapered m-potential domain. For example, if $n \geqslant 2m+1$, then $\Re^n$ is a tapered m-potential domain, but there does not exist even a positive solution s for the equation $(-\Delta)s = 1$.
(2) If Ω is a domain relatively compact in a Riemannian manifold R, then Ω is a tapered m-potential domain. For, if $(-\Delta)s = 1$ on R, then s is a bounded continuous function on Ω.

References

[1] V. Anandam, Sur une propriété de la fonction harmonique, Bull. Sci. Math. 101 (1977) 255–263.
[2] V. Anandam, Subharmonic functions outside a compact set in $\mathfrak{R}^n$, Proc. Amer. Math. Soc. 84 (1982) 52–54.
[3] V. Anandam, Biharmonic classification of harmonic spaces, Rev. Roumaine Math. Pures Appl. 45 (2000) 383–395.
[4] M. Nakai, On Evans kernel, Pacific J. Math. 22 (1967) 125–137.
[5] M. Nicolescu, Les Fonctions Polyharmoniques, Hermann, Paris, 1936.
[6] S.I. Othman, V. Anandam, Liouville–Picard theorem in harmonic spaces, Hiroshima Math. J. 28 (1998) 501–506.